Title: Matroids: The Theory and Practice of Greed

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Abstract: A matroid is a mathematical structure that generalizes the notion of linear independence. Remarkably, this simple and elegant mathematical structure can be used to systematically develop efficient and simple “greedy” algorithms for a variety of discrete optimization problems. Moreover, matroids provide some insight into why other discrete optimization problems are apparently computationally intractable. This module introduces matroids and demonstrates their application to several discrete optimization problems.

Prerequisites: The reader is assumed to be familiar with elementary concepts in linear algebra (definition and properties of linear independence) and elementary concepts in graph theory (definition of a graph, bipartite graph, and path).
Introduction

This paper shows how some natural generalizations of concepts from linear algebra can be used to find simple and efficient algorithms for many discrete optimization problems. Specifically, we describe a mathematical structure called a matroid. We then show how matroids can be used to construct so-called greedy algorithms for a variety of discrete optimization problems.

We begin by considering the case of a new long-distance telephone company. The company plans to offer service between $n$ cities but to lease the actual phone lines from existing companies. A direct phone line exists between certain pairs of cities and there is a positive cost associated with leasing each line. The new company would like to lease a subset of lines such that it can provide a path between any two cities using only the leased lines. A solution is any subset of lines with this property. An optimal solution is a solution of minimum total cost. For example, consider the weighted graph in Figure 1(a) where vertices correspond to cities, edges correspond to existing phone lines, and the edge weights correspond to the leasing costs of the phone lines. One possible solution is shown in Figure 1(b) with a total cost of 18. Another solution is shown in Figure 1(c) with a total cost of only 11. It is possible to verify that the solution in Figure 1(c) is an optimal solution.

Note that any solution must span all of the vertices in the graph and an optimal solution must be a tree: a connected graph with no cycles. If a solution contains a cycle then any edge on the cycle can be removed without destroying connectivity while decreasing the cost of the solution. Thus, given a connected graph, an optimal solution is a spanning tree of minimum total cost, also known as a minimum spanning tree.

The minimum spanning tree problem was studied as early as 1926 by the Czech mathematician Otakar Borůvka in connection with minimizing the cost of electric networks (Borůvka, 1926). The problem was later studied by Prim (Prim, 1957) and Kruskal (Kruskal, 1956) among others. Kruskal showed that the following simple algorithm can be used for finding a minimum spanning tree in a connected graph: Let $S$ be an initially empty set. Sort the edges of the graph in order of nondecreasing edge costs. Consider each edge in the sorted list, beginning with an edge of least cost, and add the edge to $S$ if and only if it does not create a cycle with the edges already in $S$. Kruskal showed that after all edges have been considered, $S$ is a minimum spanning tree for the graph. For example, in the graph in Figure 1, Kruskal’s algorithm sorts the edges in the order 1, 2, 3, 4, 5, 5, 6. It begins by selecting the edge of weight 1 and adding it to $S$. The edge of weight 2 is added next, followed by the edge of weight 3. The edge of weight 4 is considered next, but cannot be
added to \( S \) because it creates a cycle with existing edges in \( S \). Next, one of the two edges of weight 5 will be considered and added to \( S \). The next edge of weight 5 cannot be added to \( S \) because it creates a cycle with existing edges. Similarly, the edge of weight 6 cannot be added. At this point the algorithm terminates and \( S \) is a minimum spanning tree.

Kruskal's algorithm is said to be “greedy” because at each step it simply chooses the cheapest remaining edge that does not introduce a cycle. In general, a “greedy” algorithm is one that makes a sequence of locally optimal decisions. In the case of the minimum spanning tree problem, this sequence of locally optimal decisions leads to a globally optimal solution. Unfortunately, for other problems, greedy algorithms do not always find optimal solutions.

Consider the famous and seemingly similar traveling salesperson problem. In this problem we are given a graph with \( n \) vertices representing \( n \) cities. There is an edge between every pair of vertices with an associated positive real number representing the cost of a direct flight between the corresponding cities. A salesperson wishes to start at her home city, visit each city at least once, and return to her home city. However, since she is very busy, the salesperson stipulates that she will not fly to any city more than once. Thus, the salesperson wishes to find a cycle that visits each city exactly once: a traveling salesperson tour or Hamiltonian cycle in the graph. Among all tours, the salesperson wants to find one which minimizes the sum of the edge costs. For example, consider the weighted graph with five vertices, representing five cities, in Figure 2(a). One tour is shown in Figure 2(b) and has a total cost of 12. Another tour is shown in Figure 2(c) and has a cost of 7. It can be shown that the tour in Figure 2(c) is an optimal tour.

It seems intuitively natural to try to solve the traveling salesperson problem using a greedy algorithm. For example, starting at the home city, we might “greedily” select the least expensive flight leaving that city. Assume that this flight brings us to city \( v \). We could now “greedily” select the cheapest flight from city \( v \) that brings us to a city that we have not yet visited. The process can be repeated until we reach a city such that all other cities have been visited. At this point we are forced to fly directly to the starting city and the tour is complete.

Surprisingly, this greedy approach for the traveling salesperson problem does not always find optimal solutions. For example, consider the graph with four vertices in Figure 3(a). If vertex \( a \) represents the start city, the greedy algorithm selects vertex \( b \) as the next city...
Figure 2: (a) A weighted graph with five vertices. (b) A tour with cost 12. (c) An optimal tour with cost 7.

Figure 3: (a) A weighted graph with four vertices. (b) A solution with cost 1,000,003. (c) An optimal solution with cost 6.

followed by vertex $c$, then vertex $d$, and finally returning to vertex $a$. The cost of this tour, shown in Figure 3(b) is 1,000,003. On the other hand, the tour in Figure 3(c) has a cost of 6.

Not only does this particular greedy algorithm fail for the traveling salesperson problem, but no efficient algorithm is known for this problem. In fact, the traveling salesperson problem is a NP-complete problem. This means, roughly, that not only is no efficient algorithm known for the problem but that the discovery of an efficient algorithm would immediately imply efficient algorithms for a multitude of other apparently intractable computational problems.

While many discrete optimization problems can be solved by greedy algorithms, many others seem not to be amenable to greed or even any efficient algorithm at all. This article describes how an elegant mathematical structure, called a matroid, can be used to construct and establish the correctness of greedy algorithms for a variety of discrete optimization problems. Moreover, matroids also help us understand why certain other problems are not amenable to efficient solution.

In the following sections of this paper, we begin by defining the concept of a matroid.
Next, we show how matroids are directly related to greedy algorithms. We then use matroids to find greedy algorithms for several optimization problems. We conclude with a discussion of the relationship between matroids and intractable problems such as the traveling salesperson problem and give a brief overview of some other mathematical structures related to matroids and their applications.

**Matroids**

A matroid is a mathematical structure, introduced by Whitney in 1935 (Whitney, 1935), that generalizes the notion of linear independence. Recall that in any vector space, an independent set of vectors has the property that each of its subsets is also an independent set. In addition, if $X$ and $Y$ are two independent sets such that $|X| > |Y|$ then there exists some element $e \in X - Y$ such that $Y + e$ is also an independent set.\(^2\)

Remarkably, there are many sets, other than vector spaces, with their own associated definitions of “independence” that satisfy the two properties above. A matroid is any structure that satisfies these properties.

**Definition 1**  A matroid is an ordered pair $M = (E, I)$ where $E$ is a finite set and $I$ is a set of subsets of $E$ satisfying the following two properties:

**Heredity Property:** The empty set is in $I$ and for any set $X \in I$ all subsets of $X$ are also elements of $I$.

**Exchange Property:** If $X, Y \in I$ such that $|X| > |Y|$ then there exists some $e \in X - Y$ such that $Y + e \in I$.

The elements of set $I$ are called, not surprisingly, the independent sets of $M$.

As an example of a matroid, consider any matrix whose elements are real numbers. Let $E$ be the set of rows of the matrix and let $I$ be the set of all linearly independent subsets of $E$. Now, $M = (E, I)$ is easily verified to be a matroid. In fact, the name “matroid” comes from this relationship with matrices.

A more interesting example of a matroid is one induced from a graph, known as the graphic matroid. Consider a graph $G = (V, E)$ where $V$ is the set of vertices and $E$ is the set of edges. If $E' \subseteq E$ and $F = (V, E')$ contains no cycles, then $F$ is said to be a spanning forest of $G$. Each connected component in the spanning forest is called a tree. For any graph $G = (V, E)$, let $I$ be the set of all $E' \subseteq E$ such that $(V, E')$ is a spanning forest of $G$. We claim that $M_G = (E, I)$ is a matroid.

The heredity property of matroids is easily seen to hold for $M_G$: Since $(V, \emptyset)$ is a spanning forest of $G$, $\emptyset \in I$. In addition, if $E' \in I$ then $(V, E')$ is a spanning forest and thus $(V, E'')$ is a spanning forest for any $E'' \subseteq E'$.

The exchange property requires slightly more work to verify. Assume that graph $G$ has $n$ vertices and let $F = (V, E')$ be a spanning forest in $G$. Note that if $E' = \emptyset$ then it contains no

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\(^2\)Throughout this paper, the notation $|A|$ denotes the cardinality of set $A$, $A - B$ denotes the set of elements in set $A$ that are not in set $B$, and $A + e$ denotes the set formed by adding element $e$ to set $A$. 

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edges and thus comprises \( n \) distinct trees, each of which is a single vertex. If \(|E'| = 1\), then \( F \) comprises \( n - 1 \) trees: One tree is two vertices connected by an edge and the remaining \( n - 2 \) trees are distinct vertices. In general, if an edge \( e \notin E' \) does not create a cycle when added to \( E' \), then the edge must connect two distinct trees in \( F \). Therefore \( F + e \) comprises one fewer tree than does \( F \). Thus, in general, a forest with \( k < n \) edges comprises exactly \( n - k \) trees.

In relation to the graphic matroid, let \( X \) and \( Y \) be two edge sets in \( I \) such that \(|X| > |Y|\). Thus, \( F_X = (V,X) \) and \( F_Y = (V,Y) \) are both spanning forests of \( G \). By the above observation, \( F_X \) comprises fewer trees than does \( F_Y \). Therefore, there is some tree in \( F_X \) whose vertices are in more than one tree of \( F_Y \). This means that there is an edge \( e \in X \) whose endpoints are in different trees of \( F_Y \), and therefore \( F_Y + e \) does not contain a cycle and is therefore a spanning forest. Thus, \( M_G \) is indeed a matroid. For example, for the graph in Figure 1 some of the independent sets are \( \{ab\} \), \( \{ab, bc, de\} \), and \( \{ae, ce, de, bc\} \), and \( \{ab, bd, cd, de\} \).

Before turning to applications of matroids to discrete optimization problems, we note that the analogy between matroids and vector spaces does not end with independent sets. A related analogy, which will be used extensively in the next section, is that of a basis. In a vector space, a basis is a maximal independent set; an independent set such that the addition of any other vector to this set results in a set that is no longer independent. Similarly, a basis in a matroid \( M = (E,I) \) is defined to be a maximal independent set; an element \( I \in I \) such that \( I + e \notin I \) for all \( e \in E - I \). A fundamental result of vector spaces is that all bases have the same size. Analogously, we can directly apply the definition of a matroid to prove the following lemma.

**Lemma 1** All bases of a matroid have the same size.

The proof of this lemma and other analogies between vector spaces and matroids are explored in the exercises.

## From Matroids to Greedy Algorithms

We are now ready to establish the connection between matroids and greedy algorithms. In a discrete optimization problem, each element typically has some associated cost or weight and we wish to find a solution of minimum or maximum total weight. In the minimum spanning tree problem, for example, a weight is associated with each edge in the graph and we wish to find a spanning tree of minimum total weight.

For a given matroid \( M = (E,I) \), let \( w \) be a weight function that assigns a real number \( w(x) \) to each \( x \in E \). We can easily extend the definition of weight to apply to sets of elements: For any set \( S \subseteq E \), we define the weight of the set \( S \) to be

\[
w(S) = \sum_{x \in S} w(x).
\]

To see how this extension of the weight function is useful, we revisit the the minimum spanning tree problem. For a given instance of the minimum spanning tree problem we can
construct a corresponding graphic matroid $M_G = (E, I)$. Recall that $E$ is the set of edges in
the graph and $I$ is the set of all spanning forests in the graph. Let $w$ be a weight function
that assigns a positive weight to each edge in the graph. Then we claim that the objective
of the minimum spanning tree problem is that of finding a basis of $M_G$ such that $w(X)$ is
minimized. To see this, recall that a basis is a maximal independent set; an element in $I$
such that no proper superset of this element is in $I$. In this case, elements in $I$ are spanning
forests and thus a maximal element is a forest such that no edge can be added without
creating a cycle. Such a forest is a spanning tree. Thus, a basis of minimum weight in the
graphic matroid is a minimum spanning tree in the graph.

A remarkable property of matroids is that a basis of maximum weight can be found using
a simple greedy algorithm. (The case of finding a basis of minimum weight will be shown to
follow easily from this.) Given a matroid $M = (E, I)$ and a weight function $w : E \rightarrow \mathbb{R}$ the
matroid greedy algorithm performs the following steps:

Sort the $n$ elements of $E$ into list $e_1, e_2, \ldots, e_n$ such that $w(e_1) \geq w(e_2) \geq \ldots \geq w(e_n)$
Let $X = \emptyset$
for $i = 1$ to $n$
  if $X + e_i \in I$
    then let $X = X + e_i$
return $X$

**Theorem 1** For any matroid $M = (E, I)$ and weight function $w : E \rightarrow \mathbb{R}$ the matroid
greedy algorithm returns a basis of maximum weight.

**Proof:** First, we observe that the set $X$ returned by the algorithm is a basis of $M$. If not
then there exists $e_i \in E - X$ such that $X + e_i \in I$. By the heredity property, every subset
of $X + e_i$ is in $I$ and thus $e_i$ would have been added to the set $X$ in step $i$ of the for
loop, a contradiction.

Let $X = \{x_1, \ldots, x_k\}$ where $w(x_1) \geq w(x_2) \geq \ldots \geq w(x_k)$. Let $Y$ be a basis of $M$
of maximum weight. By Lemma 1, $|Y| = k$. Let $Y = \{y_1, \ldots, y_k\}$ where $w(y_1) \geq w(y_2) \geq
\ldots \geq w(y_k)$. If $w(x_i) \geq w(y_i)$ for all $i$, $1 \leq i \leq k$, then $X$ is also a basis of maximum
weight. Assume therefore that this is not the case and let $\ell$ be the least value such that
$w(x_\ell) < w(y_\ell)$. Consider the sets

$$X_{\ell-1} = \{x_1, \ldots, x_{\ell-1}\} \text{ and } Y_\ell = \{y_1, \ldots, y_\ell\}.$$ 

By the heredity property these sets are independent. The exchange property implies that
there exists some $y_\ell \in Y_\ell - X_{\ell-1}$ such that $X_{\ell-1} + y_\ell \in I$. Since $w(y_\ell) \geq w(y_\ell) > w(x_\ell)$, the
greedy algorithm considers $y_\ell$ before $x_\ell$. By the heredity property, every subset of $X_{\ell-1} + y_\ell$
is in $I$ and thus the algorithm would have included $y_\ell$ in $X$, a contradiction. □

As an example, we now revisit the minimum spanning tree problem. As discussed earlier,
the minimum spanning tree problem is exactly that of finding a basis of minimum weight in
the graphic matroid $M_G = (E, I)$. Assume for a moment that we actually wanted to find
a spanning tree of maximum weight; a maximum spanning tree. In this case, Theorem 1
tells us that we can simply apply the matroid greedy algorithm to the corresponding graphic
matroid. In other words, we begin by sorting the edges in $E$ in order of nonincreasing
weights. Beginning with an initially empty set $X$, we consider the sorted edges in $E$ one at a time. If an edge under consideration does not create a cycle with respect to the edges already in $X$, we add the edge to $X$. This results in a basis of maximum weight which is a maximum spanning tree. Now observe that a minimum spanning tree can be found by replacing each $w(e)$ in the graph by its negative, $-w(e)$, and finding a maximum spanning tree in this reweighted graph. Note that this is exactly equivalent to the aforementioned algorithm due to Kruskal.

In the next two sections we explore two more discrete optimization problems. For each of these problems, we find a corresponding matroid. We can then apply the matroid greedy algorithm to solve each of these problems.

**A Scheduling Problem**

Consider a set $S = \{1, \ldots, n\}$ of $n$ jobs that must be performed by a single machine. Each job takes one unit of time to complete and once a job is started on the machine it must be completed before the next job can be started. For each job there is a deadline $d(i)$ such that $1 \leq d(i) \leq n$, $1 \leq i \leq n$. For each job there is also an associated positive real reward or weight, $w(i)$, that is obtained if the job is completed no later than its deadline. The objective is to determine a schedule, an ordering of the $n$ jobs, that maximizes the total reward. In practice, these jobs might be mechanical tasks performed on a machine or programs run on a computer and the rewards might represent profits earned by completing the jobs by their deadlines. We solve this problem by constructing a corresponding matroid, showing that an optimal solution to the scheduling problem corresponds to a basis of maximum weight in the matroid, and then finding such a basis using the matroid greedy algorithm.

Given a schedule for the $n$ jobs, we say that a job is on time if it is completed on or before its deadline and otherwise the job is said to be late. Let $S$ be a set of jobs with associated deadlines and weights. A feasible schedule for $X \subseteq S$ is a schedule in which all of the jobs in $X$ are on time. A subset $X$ of $S$ is said to be feasible if there exists at least one feasible schedule for $X$. Notice that the problem of finding an optimal schedule for $S$ can be reduced to that of finding a feasible subset $X \subseteq S$ of maximum total weight. We would then like to find a feasible schedule for $X$. The elements in $S - X$ will all be late and can therefore be scheduled in any order after the jobs in $X$. Of course, we will need some way of determining whether a subset $X \subseteq S$ is feasible and, if it is, we will need to find a feasible schedule for it.

Imagine for a moment that we are given a set $X \subseteq S$ and told that $X$ is feasible. We know that a feasible schedule exists for $X$, but it would seem that we might need to test all of the permutations of the elements of $X$, one by one, until we find a feasible schedule. This, of course, would be a prohibitively slow process. Surprisingly, if $X$ is known to be feasible, it is very easy to find a feasible schedule: we simply sort the jobs in order of nondecreasing deadlines. Perhaps even more remarkable is the fact that to determine whether or not a set $X$ is feasible in the first place, we need only sort the jobs in order of nondecreasing deadlines and check to see if every job is on time in this schedule. We now formalize these claims in the following lemma.

**Lemma 2** A set $X$ is feasible if and only if the schedule formed by sorting the elements of
X in order of nondecreasing deadlines results in each job being on time.

**Proof:** Assume that X is feasible. Then there exists some schedule that completes the jobs in X on time. If this schedule has some pair of jobs i and j both of which are completed on time but such that i is completed before j and \( d(i) > d(j) \), then we swap the two jobs. Now, job j is completed even earlier and is therefore still on time. Moreover, job i is now completed when j was completed in the original schedule and is therefore completed no later than time \( d(j) \). Since \( d(i) > d(j) \), job i is still on time in this new schedule. Therefore, given any schedule that completes all of the jobs in X on time, we can repeatedly swap pairs of on-time jobs until they are completed in order of nondecreasing deadlines with all jobs still being completed on time.

Conversely, assume that for a given set X, the schedule formed by sorting the elements of X in order of nondecreasing deadlines results in each job being on time. Since this is a feasible schedule for X, X is feasible by definition.

Let us define \( I \) to be the set of all feasible subsets of S. We will next show that \( M_S = (S, I) \) is a matroid. Since all weights are positive in this problem, a feasible set of maximum weight is precisely a basis of maximum weight in the matroid. Thus, the matroid greedy algorithm can be applied to solve this scheduling problem.

**Lemma 3** Given a set of jobs S, let \( I \) denote the set of all feasible subsets of S. Then \( M_S = (S, I) \) is a matroid.

**Proof:** The heredity property is satisfied because \( \emptyset \) is trivially feasible and every subset of a feasible set is clearly also feasible.

To show that the exchange property is satisfied, let X and Y be two elements of I such that |X| > |Y|. Without loss of generality, assume that |X| = |Y| + 1. (If this is not the case, we simply remove elements from X arbitrarily until this assumption is true.) Let |Y| = n and let \( x_1, \ldots, x_{n+1} \) and \( y_1, \ldots, y_n \) denote the elements of X and Y, respectively, in order of nondecreasing deadlines. That is, \( d(x_i) \leq d(x_{i+1}) \), \( 1 \leq i \leq n \), and \( d(y_j) \leq d(y_{j+1}) \), \( 1 \leq j \leq n-1 \). If \( x_{n+1} \notin Y \) then \( x_{n+1} \in X - Y \) and the schedule \( y_1, \ldots, y_n, x_{n+1} \) is a feasible schedule for \( Y + x_{n+1} \) because \( x_{n+1} \) completes at time \( n + 1 \) in the schedule for X and thus \( d(x_{n+1}) \geq n + 1 \).

Assume, therefore, that \( x_{n+1} \in Y \). Since |X| > |Y|, there exists some element of X that is not in Y. Let k be the largest value of i such that \( x_i \notin Y \). Then \( 1 \leq k \leq n \) and \( x_k \notin Y \) but \( x_j \in Y \) for \( k < j \leq n + 1 \). Since \( x_{k+1}, \ldots, x_{n+1} \in Y \) and the elements \( x_1, \ldots, x_{n+1} \) and \( y_1, \ldots, y_n \) both appear in order of nondecreasing deadlines, it must be the case that \( d(y_n) \geq d(x_{n+1}) \) and, in general, \( d(y_i) \geq d(x_{i+1}) \), \( k \leq i \leq n \). Also, \( d(x_i) \geq i \), \( 1 \leq i \leq n + 1 \). Thus, \( d(y_i) \geq d(x_{i+1}) \geq i + 1 \), \( k \leq i \leq n \). This implies that in the schedule \( y_1, \ldots, y_n \) we can “shift” the elements \( y_k, \ldots, y_n \) so that they now complete at times \( k + 1, \ldots, n + 1 \), respectively and are all still on time. This leaves a “gap” at time k. Since \( x_k \notin Y \) and \( d(x_k) \geq k \), we move \( x_k \) into this gap. We now have the set \( Y + x_k \) with feasible schedule \( y_1, \ldots, y_{k-1}, x_k, y_k, \ldots, y_n \). Therefore, \( Y + x_k \) is a feasible set and the exchange property is satisfied.

Since we have shown that \( M_S \) is a matroid and that the scheduling problem can be formulated as that of finding a basis of maximum weight in this matroid, the matroid greedy
algorithm can be applied to solve this problem. Specifically, the matroid greedy algorithm begins by sorting the jobs in order of nonincreasing weights. The set $X$ is initially empty. Each job $e_i$ is considered according to the sorted order and is added to $X$ if and only if $X + e_i$ is independent. To test if $X + e_i$ is independent, all the jobs in $X + e_i$ are sorted in order of nondecreasing deadlines. Each job in this sorted order is then checked to determine if it completes by its deadline. If all the jobs are completed by their deadlines, $X + e_i$ is independent and $e_i$ is added to $X$. Otherwise, the algorithm does not add $e_i$ to $X$. When all jobs have been considered, the set $X$ is a feasible set of maximum total weight. To find an optimal schedule, we simply sort the elements in $X$ in order of nondecreasing deadlines. We then append the elements in $S - X$ in arbitrary order to the end of this schedule. Exercise 8 provides a small example on which the algorithm may be performed.

A Task Assignment Problem

A company has $m$ employees $E = \{e_1, \ldots, e_m\}$ and $n$ tasks $T = \{t_1, \ldots, t_n\}$ that must be completed. Each employee is qualified to perform a certain subset of these tasks but has time to perform at most one task. There is a positive real weight $w(t_i)$ associated with each task, representing the value or priority of that task. The objective is to find an assignment of tasks to employees, where each task is assigned to at most one employee and each employee is assigned to at most one task, such that the value of the completed tasks is maximized.

This optimization problem can be modelled with a bipartite graph in which there are two vertex sets $E = \{e_1, \ldots, e_m\}$ and $T = \{t_1, \ldots, t_n\}$. There is an edge from a vertex $e_i$ to a vertex $t_j$ if employee $e_i$ is qualified to perform task $t_j$. A matching in the graph is a subset of edges such that no two edges share a common vertex. For a given matching, a vertex is said to be matched if some edge in the matching is incident on it. Our objective then is to find a matching in the graph that maximizes the sum of the weights of the matched vertices in $T$.

This problem too can be solved with a greedy algorithm. To do so we begin by defining a matroid known in the literature as a transversal matroid. Given a bipartite graph with vertices $E$ and $T$ we define $X \subseteq T$ to be matchable if there exists some matching in the graph that matches every vertex in $X$ to some vertex in $E$. Let $\mathcal{I}$ denote the set of all matchable subsets of $T$. We claim that $M_T = (T, \mathcal{I})$ is a matroid. Notice that an optimal solution to our problem is exactly that of finding an independent set of maximum weight in the matroid. Since all tasks in $T$ have positive weight, an independent set of maximum weight is a basis of maximum weight. Thus, by showing that $M_T$ is a matroid we will be able to solve the optimization problem using the matroid greedy algorithm.

**Lemma 4** Given a bipartite graph with vertices $E \cup T$, let $\mathcal{I}$ denote the set of all matchable subsets of $T$. Then $M_T = (T, \mathcal{I})$ is a matroid.

**Proof:** The heredity property is satisfied because $\emptyset$ is trivially matchable and any subset of a matchable set is clearly also matchable.

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3The technical name for such a subset is a partial transversal.
To show that the exchange property is satisfied, let $X$ and $Y$ be two elements of $I$ such that $|X| > |Y|$. Since $X$ and $Y$ are matchable, let $M_X$ and $M_Y$ be matchings that match the vertices in $X$ and $Y$, respectively, with vertices in $E$. Color the edges of $M_X - M_Y$ black, the edges of $M_Y - M_X$ white, and the edges of $M_X \cap M_Y$ gray. Notice that every edge in $M_X \cup M_Y$ is colored black, white, or gray. Since the number of edges in $M_X$ is exactly equal to the number of vertices in $X$, and similarly for $M_Y$ and $Y$, $|M_X| > |M_Y|$. This implies that there are more black edges than white edges.

Next, consider the subgraph $M$ induced by the black and white edges.\textsuperscript{4} Observe that each vertex in $M$ is incident on at most two edges in $M$ because at most it is incident on one edge in $M_X$ and one edge in $M_Y$. Therefore, the vertices of $M$ have degree one or two. This implies that $M$ comprises only cycles and paths. Because no vertex of $M$ can be incident on two edges of the same color, the edges on these cycles and paths alternate between black and white; these are called alternating cycles and paths. Each alternating cycle has an equal number of black and white edges. Since there are more black edges than white edges, some alternating path must have more black edges than white edges.

Let $P$ be an alternating path with more black edges than white edges. Let $v_1, v_2, \ldots, v_k$ denote the vertices on path $P$ from one endpoint to the other. Since $P$ has more black edges than white edges, the first and last edges on this path must be black. Therefore, the path has an odd number of edges and thus an even number of vertices. Since the graph is bipartite, the vertices alternate between being in $E$ and $T$. This means that one endpoint of $P$ is in $E$ and the other is in $T$. Without loss of generality, assume that $v_1 \in T$.

We now claim that $v_1 \in X - Y$. First, $v_1 \in X$ because it is incident on an edge in $M_X$. Assume, by way of contradiction, that $v_1 \in Y$. Then $M_Y$ must contain an edge incident on $v_1$. Such an edge is either white or gray. The black edge in $P$ incident on $v_1$ is, by definition, from matching $M_X$. Since there cannot be a second edge of $M_X$ incident on $v_1$, no gray edge is incident on $v_1$. If there is a white edge incident on $v_1$ then this edge is part of $P$, contradicting the assumption that $v_1$ is an endpoint of $P$. Thus, $v_1 \in X - Y$.

Since $v_1 \in X - Y$, we now consider the set $Y + v_1$. To demonstrate that $Y + v_1$ is matchable, consider the matching $M_Y$ modified by removing the white edges in $P$ from $M_Y$ and adding the black edges in $P$ to $M_Y$. This set is still a matching and, in addition to matching every vertex in $Y$, matches $v_1$ as well. Thus $Y + v_1$ is matchable and is therefore an element of $I$. \hfill \square

Now that we have a matroid corresponding to this matching problem, we can use the matroid greedy algorithm to determine an optimal solution: for any graph, we first sort the vertices in $T$ in nonincreasing order of weight and then we repeatedly choose a maximum weight vertex that maintains independence. The problem of determining if a set is independent in this context, that is whether a subset of $T$ is matchable, can be solved using the bipartite matching algorithm often covered in a graph theory course (Bondy & Murty, 1976) or network flow algorithms generally covered in an algorithms course (Cormen et al., 1990). Exercise 9 provides an example on which the greedy algorithm may be performed. This example is sufficiently small that testing for independence can easily be performed by inspection.

\textsuperscript{4}The subgraph \textit{induced} by the black and white edges is the graph formed by considering only those edges and the vertices incident to them.
Conclusion

We have seen that matroids are a powerful tool for constructing and showing the correctness of greedy algorithms. Matroids can also be used to develop efficient algorithms for even more difficult optimization problems where simple greedy algorithms fail. For example, consider a variant of the bipartite matching problem that arose in the task assignment problem. In that problem, weights were associated with a subset of the vertices in the graph and the objective was to find a matching that maximized the sum of these weights. Now consider the situation in which there is a positive weight associated with each edge, rather than with some vertices, and we wish to find a matching that maximizes the sum of the weights on the matched edges. Although the greedy algorithm does not always find an optimal solution for this problem, the problem can be solved using the notion of matroid intersections.

Given two matroids $M_1 = (E, I_1)$ and $M_2 = (E, I_2)$ over the same set $E$ and a weight function $w : E \to \mathbb{R}$, the matroid intersection problem for two matroids is that of finding an element $X \in I_1 \cap I_2$ of maximum weight. Efficient algorithms are known for solving the matroid intersection problem for two matroids. Many discrete optimization problems, including the aforementioned bipartite matching problem with edge weights, can be formulated as a matroid intersection problem for two matroids.

Some optimization problems that cannot be solved by greedy algorithms or even using the intersection of two matroids can be formulated as the intersection of three or more matroids. The traveling salesperson problem, for example, can be formulated as that of finding an independent set of maximum weight in three matroids. Unfortunately, the matroid intersection problem for three or more matroids is NP-complete.

Several mathematical structures related to matroids have also been studied. For example, an antimatroid is a structure with a weaker version of the heredity property but a stronger version of the exchange property. A common framework for matroids and antimatroids is a structure called a greedoid (Korte et al., 1991), which uses the weaker versions of both the heredity and the exchange properties. These two structures have applications in the development of greedy algorithms for optimization problems as well.

Finally, while we have investigated applications of matroids to discrete optimization, matroids and their related structures have a variety of other applications in mathematics. Active areas of current research are in applications of such structures in algebra, geometry, and topology.

We refer the interested reader to several excellent books. Oxley’s text provides a comprehensive introduction to matroid theory. Texts by Lawler and by Papadimitriou and Steiglitz discuss the applications of matroids to discrete optimization.

References
